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Demonstration of energy dissipation in a spring-mass system undergoing free oscillations in air

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ABSTRACT: In this article, an experimental method that uses logarithmic decrements to demonstrate energy dissipation in a spring-mass system that is oscillating in a medium with friction is presented. Such a method can be used to calculate, as a function of time, the amount by which frictional forces dissipate the kinetic energy of the oscillating mass. This information is useful in determining whether the applied force of damping is nonlinear or linear, whether or not it changes from the former to the latter during motion, and when such a change occurs. The method is explained and illustrated by using three spring-mass systems that oscillate in air with circular frequencies that vary between 9 and 25 rad/s.

INTRODUCTION

In modelling the damped oscillations of a spring-mass system, it is customary to represent the effects of fluid friction by introducing the concept of viscous damping, a drag force that is proportional to the first power of the velocity with the coefficient of proportionality, referred to as the coefficient of (viscous) damping, being presumed constant. This model leads to a second-order linear ordinary differential equation with constant coefficients and, due to its simplicity, perhaps, it is widely used in the mathematical modelling of dynamic systems [1-4].

A review of the literature shows that analytical results based on the viscous-damping model do not always agree with experimental data [5-10]. There are three common areas of discrepancies. Damped harmonic motion leads one to expect the amplitude of oscillation to decay exponentially with time; however, this is rarely observed in the laboratory. Indeed, while exponential decay can be achieved, such is the case only in special and carefully controlled circumstances [5]. Second, in the vast majority of experiments, the rate of decay with time of the amplitude of oscillations not only does not follow what exponential decay indicates but its departure from the indicated behaviour is particularly high in the early stages of motion and it decreases with the duration of motion [6]. Finally, attempts to use this model in the laboratory to determine the viscosity of the fluid in which the mass is oscillating have proved to be very inaccurate; to obtain usable results, investigators had to introduce correction factors [7].

When experimental data deviate significantly from the results of the viscous-damping model, it is rightly concluded that the force that was responsible for damping was not directly proportional to the first power of the velocity, but it is difficult to infer the exact mathematical nature of this force from experiments alone [8]. Therefore, it is generally determined that the experiment was a failure [6-11]. However, whether experimental data agree exactly with, or deviate significantly from, the results of the viscous-damping model, they can be used to examine how energy was dissipated with time during oscillations, thereby illustrating how the work done by the applied force of friction varied during the motion. The purpose of this article is to present the results of a series of experiments that were used successfully to demonstrate how this can be done.

The remainder of the article is organised in the following manner: first, the viscous-damping model is presented. Then, the concept of the logarithmic decrement is introduced and related to energy dissipation. After that, the strategy that was used to determine logarithmic decrements is explained and this is followed by a brief description of three experiments and a presentation of their results. Next, experimental results are analysed, discussed, and related to viscous-damping theory. Finally, general conclusions are stated.

THE VISCOUS-DAMPING MODEL

We consider a linear spring of stiffness k that is arranged vertically. The upper end of this spring is fixed while the lower end is attached to an object of mass m. The whole assembly hangs in the open air of laboratory far from any walls. The mass is set into motion by an appropriate combination of an initial displacement x_0, and an initial velocity v_0. Thereafter, it oscillates freely in air that is otherwise quiescent.

If we assume viscous damping a priori, that is, that the resistance to motion by the surrounding air is equivalent to a force F, that is directly proportional to the first power of the velocity of the oscillating mass, such that $F = C\dot{x}$, then, the differential equation of the ensuing motion is given by [4]:

$$\ddot{x} + 2\varsigma\dot{x} + \omega_n^2 x = 0, \tag{1}$$

with the initial conditions

$$\begin{aligned} x(t=0) &= x_0 \\ \dot{x}(t=0) &= v_0 \end{aligned}, \tag{2}$$

where x is the instantaneous position of the mass, measured from its static equilibrium; the dots indicate derivatives of x with respect to time t; ω_n is the natural frequency of oscillation of the system in vacuum. It is given by

$\omega_n = \sqrt{\dfrac{k}{m}}$; and ς , the damping ratio, is related to the damping coefficient, C, by $\varsigma = \dfrac{C}{2m\omega_n}$.

When $0 < \varsigma < 1$, motion is underdamped; when $\varsigma > 1$, it is overdamped; and when $\varsigma = 1$, the motion is critically damped. Oscillations are possible only when the motion is underdamped, which is our focus in this work.

When motion is underdamped, the solution to Equation (1), subject to the initial conditions in Equation (2), is given by [4],

$$x(t) = X_0 e^{-\varsigma\omega_n t}\cos(\omega_d t - \phi_0), \tag{3}$$

where,

$$\begin{aligned} \omega_d &= \sqrt{1-\varsigma^2}\,\omega_n, \\ X_0 &= \sqrt{x_0^2 + \left(\frac{v_0 + \varsigma\omega_n x_0}{\omega_d}\right)^2}, \\ \phi_0 &= \tan^{-1}\left(-\frac{v_0 + \varsigma\omega_n x_0}{x_0\omega_d}\right). \end{aligned} \tag{3a}$$

Equation (3) indicates that the amplitude of the motion decreases exponentially with time. In geometric terms, the exponential function represents the envelope of the plot of the position of the mass with time. We used this important feature to process experimental data: we recorded the instantaneous positions of the mass with time using different spring-mass systems; and, in each case, created a plot of position versus time, extracted the envelope of the oscillations from that plot, and, subsequently, fit an exponential function to the resulting envelope. The end results verified what is reported in the literature: the fit is excellent only in special cases [11-14]. But we used the logarithmic decrement of each motion to determine the pattern of energy dissipation generated by the applied damping force.

LOGARITHMIC DECREMENTS AND ENERGY DISSIPATION

The logarithmic decrement δ, allows one to extract the damping ratio from the envelope of the plot of position versus time of a given spring-mass system. Using Equation (3), the ratio between two positions, x_i and $x_{(i+j)}$, taken by the mass at different times t_i and $t_{(i+j)}$, respectively, is given by

$$\frac{x_i}{x_{(i+j)}} = \frac{Cos(\omega_d t_i - \phi_0)}{Cos(\omega_d t_{(i+j)} - \phi_0)} e^{\zeta\omega_n(t_{(i+j)} - t_i)} \tag{4}$$

If these two instants of time are chosen in such a way that they are separated by either one period of oscillation, or by an integral multiple of that period, j, then, the cosine terms will cancel each other out of Equation (4) and the ratio of positions becomes:

$$\frac{x_i}{x_{(i+j)}} = e^{j(\frac{2\pi}{\omega_d})\zeta\omega_n} \tag{5}$$

Then, taking the natural logarithms of both sides of Equation (5) and rearranging terms gives:

$$\delta = \frac{1}{j} ln\left(\frac{x_i}{x_{(i+j)}}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{6}$$

Where δ is the logarithmic decrement, j the number of cycles, x_i the position of the mass at time t_i, and $x_{(i+j)}$ its position at time $t_{(i+j)}$.

The logarithmic decrement δ is related to the energy dissipated by the damping force per cycle. The work, ΔW, done by the damping force $F = C\dot{x}$ during one complete cycle of oscillation is determined from

$$\Delta W = \int_0^{\frac{2\pi}{\omega_d}} C\dot{x}^2\, dt \tag{7}$$

When damping is slight, $\varsigma \ll 1$, then the amplitude of oscillation decays very slowly. Under these circumstances, the work done by the damping force per cycle is approximately equal to:

$$\Delta W_0 = \pi C \omega_d x_0^2 \tag{8}$$

After writing the damping coefficient *C* in terms of the damping ratio, ς, and rearranging terms, Equation (8) becomes

$$\frac{\Delta W_0}{\left(\frac{k x_0^2}{2}\right)} = 4\pi\varsigma\sqrt{1-\varsigma^2} \tag{9}$$

Combining Equation (6) and Equation (9), it follows that, when $\varsigma \ll 1$,

$$\delta \approx \frac{1}{2}\frac{\Delta W_0}{\left(\frac{k x_0^2}{2}\right)} \tag{10}$$

It can be seen from Equation (10), therefore, that, when damping is slight, the logarithmic decrement can be used as a direct measure of the energy dissipated per cycle of oscillation of the mass. This was done below to demonstrate how data from an oscillating spring-mass system can be used to visualise how the damping force, which is applied to it, dissipates energy from the moving mass. To do this, one need not know the mathematical description of the damping force itself.

SERIES OF LOGARITHMIC DECREMENTS

As indicated in Equation (6), when motion is viscously damped, the logarithmic decrement δ is a constant that it is independent of the points used to compute it. Indeed, it is solely a function of the damping ratio of the system. Consequently, when motion is truly viscously damped, the pair of points chosen to compute δ does not affect the end result. In a typical experiment, one has many pairs of points in the same data set from which to choose. Thus, one could compute the logarithmic decrement of the corresponding motion using different pairs of points. When this was done, however, it was found that in most cases, the value of δ that was obtained from experimental data varied with the points that were used to compute it. Therefore, in practice, logarithmic decrements may depend on the points used for their computation. These variations were used to determine the departure of a given motion from that which is indicated by the viscous-damping model.

Several spring-mass systems were tested and for each set of data, many pairs of points were used to compute logarithmic decrements experimentally. Specifically, a logarithmic decrement, δ_{ij}, was computed by using the time segment from time t_i to time $t_{(i+j)}$. To get a new time segment, *i*, the starting point, was held fixed but increased the length of the most recent segment by adding a constant increment to it. Comparing the resulting values of the logarithmic decrements to each other allowed examining the extent to which they varied during the motion.

EXPERIMENTS AND RESULTS

A load cell was designed and used with computer data-acquisition equipment to collect data on a variety of objects. The test setup, equipment, and procedures pertaining to these experiments were detailed in earlier work [5]. Three different spherical masses were used. A linear spring of stiffness k= 40.68 N/m was connected to each sphere and tested; for each spring-mass system, data were recorded for 500 consecutive seconds and the envelope of the plot of the corresponding oscillations was sampled approximately every 25 seconds. Thus, in the end, the experimental envelope was represented by the twenty points sampled from it. This process was repeated for each spring-mass system. Although the duration of data collection was the same for each system, the number of cycles covered was different in each case, owing to the different periods of oscillation of the systems. The data pertaining to the tested spheres, the linear spring, and the number of cycles recorded in each case are shown on Table 1, while the envelopes of the positions of the spheres with time are shown in Figure 1. Only the first quadrant is shown in those plots, because it was verified experimentally that the shape of the whole envelope was symmetric with respect to the time (horizontal) axis.

For each sphere, a series of logarithmic decrements was computed, using the first peak (i=1) as the base (fixed reference point) and increased the number of cycles considered in each new computation by a constant increment, j that equalled the number of cycles of that oscillation that fit into 25 seconds, as shown in Table 1.

Table 1: Data on the spheres, the spring and the cycles of oscillation.

Type of object	Diameter (m)	Mass (kg)	Natural frequency (rad/s)	Cycles recorded	Cycles/segment (j)
Metal sphere	0.05	0.502	9.002	~ 717	~ 36
Softball	0.093	0.189	14.671	~1168	~ 59
Tennis ball	0.08	0.067	24.641	~1961	~ 98
Linear spring	0.006	0.003	-	-	-

Thus, for the metal sphere, j = 36; for the softball, j = 59; and for the tennis ball, j= 98, approximately. This process was continued until all the data had been used. It was necessary to round off the number of cycles because the sampling rate and the frequency of a particular spring-mass system did not always match exactly. Using the same time increment in all tests allowed the data that were collected from each sphere to yield twenty logarithmic decrements. The resulting decrements are plotted against the time segments over which they were measured in Figure 2. The decrements were normalised before plotting. Normalisation was achieved by dividing each value in a set by the arithmetic average of all the values in that set. The reason for proceeding in this way is that the viscous-damping hypothesis leads to the conclusion that all logarithmic decrements from a given oscillation should be identical, regardless of the points used to compute them. In practice, however, the magnitude of this value is not known a priori. Indeed, even the expressions proposed in the literature for estimating it are problematic [7-8]. In our analyses, therefore, the arithmetic average of all experimental decrements from a given spring mass system was used as an approximation for this hypothetical value.

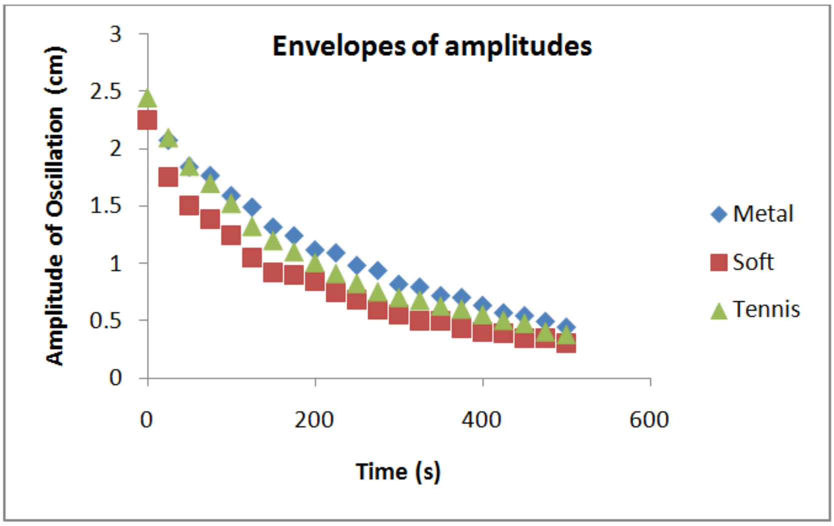

Figure 1: Envelopes of the amplitudes of oscillation of the three spheres described in Table 1.

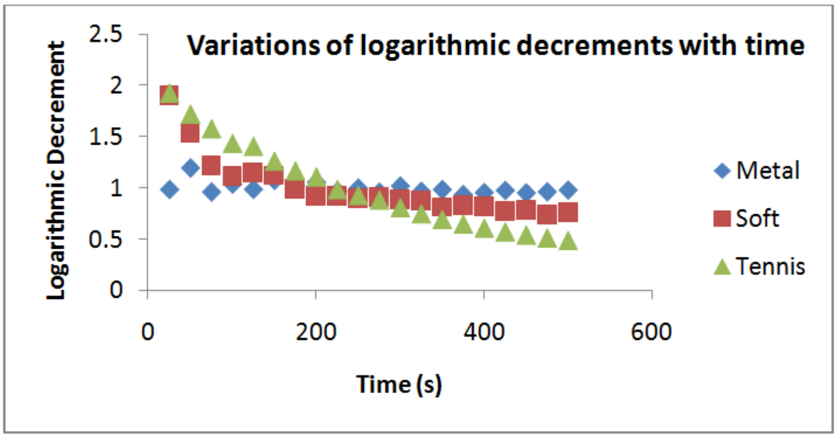

Figure 2: Variations with time of the log decrements of the three spheres described in Table 1.

DISCUSSION

Each set of data shown in Figure 1 was fitted with the best exponential function. The numerical results are shown in Table 2. It can be seen that the data from the metal sphere generated a better fit than either those from the softball, or from the tennis ball. Similarly, the graphs of Figure 2 show that the logarithmic decrements of the metal sphere generated much less variability about the average value, here unity, than those from either the softball, or from the tennis ball. Since, in Equation (10), it was shown that the logarithmic decrement may be used as an indicator of energy dissipation, the data in Figure 2 demonstrate how energy was dissipated by fluid friction during the oscillation of the tested spheres. It is evident that dissipation varied with time for each sphere; but it did so in different ways. For the metal sphere, which oscillated at 9 rad/s, energy dissipation fluctuated about the arithmetic mean of the data set (by ± 10 %) and started doing so shortly after motion began; the fluctuations decreased with time and became very small (± 3 %) after a short while (72 cycles). It can be concluded that shortly after the onset of the motion, energy was essentially dissipated by the same amount from one cycle to the next for the remainder of the motion of the metal sphere. In the case of the softball, which oscillated at 14.7 rad/s, energy dissipation decreased with time almost monotonically for a longer period than in the case of the metal sphere (826 cycles). After that, it appeared to be approaching a level about which it could fluctuate. The case of the tennis ball, which oscillated at 24.64 rad/s, showed energy dissipation that decreased monotonically for a longer time yet. And it just appeared to be approaching a constant level at the end of the recorded data (1960 cycles). Unfortunately, it was not possible to collect useful data for longer than the 500 seconds allotted to the process. It was observed that, past this point, the amplitude of oscillation of the tennis ball became so small that the resolution of the data-collection process did not produce any meaningful data beyond those that were retained for processing. Indeed, it became difficult to distinguish data from noise and random perturbations.

Table 2: Results from curvefitting an exponential function to each experimental envelope.

Type of sphere	Amplitude (data)	Amplitude (fit)	Discrepancy (%)	Exponent (fit)	R^2
Metal	2.25	2.196	-2.4	-0.0031	0.997
Softball	2.25	1.807	-19.68	-0.0037	0.982
Tennis ball	2.45	2.157	-11.96	-0.0035	0.989

The use of logarithmic decrements to interpret data from the oscillations of spheres in air has clearly demonstrated what is indicated in the fluid mechanics literature on the damping of an oscillating sphere: energy dissipation due to viscosity is considerably more complicated than is commonly suggested by the viscous-damping hypothesis [15-23]. The obtained data indicate that *impulsive* start of the motion of the oscillating mass causes the coefficient of damping to be large at the beginning and to decrease with time, which is consistent with what is reported in the literature [15-21]. This delays the manifestation of viscous-damping behaviour and may even inhibit it altogether, in some cases [5-8]. The results also suggest that the higher the frequency of the oscillations, the longer the delay and, hence, the less likely it is that one would observe viscous-damping behaviour everywhere along the motion. The data pointed to the possibility of the existence of a critical length of time, or transition time (and corresponding critical amplitude ratio), that must elapse (be reached) after the start of the motion, before viscous damping becomes evident [5-6]. During transition time, the damping force is a nonlinear function of the velocity. However, as the amplitude of oscillation decreases with time, so does the linear velocity of the mass, and the damping force eventually becomes a linear function of velocity, thereby, allowing one to observe behaviour that is viscously damped. This is expected to happen, provided that energy is dissipated slowly enough from the system for it to still be oscillating after these critical values have been reached.

It is well known that the manifestation of linear damping in the motion of a sphere that is oscillating in a Newtonian fluid requires both a critical penetration depth of vorticity and a critical Reynolds number [25-26]. However, the magnitudes of these numbers are not known at this time, because their existence is predicted from the order-of-magnitude arguments that are used to both linearise the Navier-Stokes equations and to require viscous boundary layers that are thin [25-26]. For this reason, the transition time was estimated that was required for viscous damping to become noticeable from the obtained data; and it appears to be well correlated with the frequency of oscillation of the masses that were used in the tests. For the metal sphere, transition took 72 cycles; for the soft ball, it took 826 cycles; and for the tennis ball it was estimated to be beyond 1960 cycles. When estimated transition times were plotted against the corresponding frequencies, it became clear that the duration of transition increased with the frequency of oscillation and the resulting graph suggested the existence of a cut-off frequency below which viscous-damping behaviour would be readily observable for solid spheres oscillating freely in air. From the obtained data that frequency was below 8 rad/s. Accordingly, these results indicate that using solid spheres to demonstrate viscous damping in air requires choosing a frequency that is very low. Specifically, current data indicate that frequencies that fall below 8 rad/s are likely to work well. Finally, it is tempting to use R^2, the so-called goodness of fit of the envelope of amplitudes to an exponential function, as an indicator of the presence of viscous damping. However, the results suggest that this indicator is reliable only when R^2 approaches unity. Indeed, as can be seen in Table 2, the values of R^2 in all three cases shown here are very high; yet, only the metal sphere displays viscous damping almost throughout all of its oscillations. It was found in the experiments that, when the goodness of fit was combined with the knowledge of how energy was dissipated during oscillations, one learned more about the nature of the damping force than when that correlation was used in isolation.

CONCLUSIONS

In teaching and in research, it is standard practice to model the drag force on an object that is oscillating in a fluid medium as being proportional to the first power of the velocity, with a constant coefficient of proportionality [11-14]. However, assessments of this model in the laboratory have shown that it is not always accurate, indicating that damping forces due to fluid friction are nonlinear in most cases. In this article, a method was presented that has been used successfully to illustrate how energy is dissipated with time in a spring-mass system without knowing the mathematical expression for the applied drag force. The resulting patterns of energy dissipation indicate that the oscillatory motion of a sphere in viscous fluid is typically subjected to nonlinear damping forces in the first part of motion but that, as time passes, the damping force can become linear, if damping is slight enough and the frequency of oscillation is sufficiently low. The method can, therefore, be used to assess the extent to which the motion being investigated agrees with the viscous-damping model that is commonly used in engineering and physics textbooks. Furthermore, the method holds the potential to be used to determine the range of frequencies over which the damping force is likely to be linear throughout the entire motion of a spring-mass system. This method has been used successfully in three different courses: in a dynamics class, to discuss the role of the work done by the force due to fluid friction in dissipating the kinetic energy of a moving mass [24]; in a fluid mechanics class, to illustrate the role of vorticity transport from the boundary of a solid into the interior of the fluid in unsteady boundary-layer theory [25-26]; and in vibration analysis, to discuss both the practical applications and the limitations of the viscous-damping hypothesis [27-29].

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